Solution to Challenge Problem 1 in the framework of sets of probability measures

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Problem: Propagation of uncertainty through a real valued, continuous function

$$y = G(a, b)$$

of two real parameters a, b. Specification of uncertainty of a, b by: intervals, sets of intervals, probability distributions, sets of probability distributions, or a combination thereof.

A-priori choice of interpretation:

"sets of intervals with equally credible sources"
= "random set with uniform weights".

Our appoach: Interpret all types of uncertainties of the input data as sets of probablity measures.

Output: lower and upper probabilities

$$\underline{P}(y \in C) \le \overline{P}(y \in C)$$

for arbitrary Borel sets $C \subset \mathbb{R}$.

The univariate case

Hypotheses: $G : \mathbb{R} \to \mathbb{R}$ is a continuous function, $y = G(a)$; C is a Borel subset of $\mathbb{R}$.
Consistency: $\underline{P}, \overline{P}$ coincide with the usual interpretation – endpoints of intervals, belief and plausibility, necessity and possibility.

Intervals: The uncertainty of a is specified by a closed interval $A \subset \mathbb{R}$. Then

$$G(A) = [\inf\{G(x) : x \in A\}, \sup\{G(x) : x \in A\}] .$$

The upper probability is

$$\widetilde{P}(y \in C) = \begin{cases} 1, & G(A) \cap C \neq \varnothing \\ 0, & G(A) \cap C = \varnothing . \end{cases}$$

Interpreting A as the set $\mathcal{M}$ of all probability measures on A, the corresponding upper probability is

$$\begin{aligned} \overline{P}(y \in C) &= \sup\{p\big(G^{-1}(C)\big) : p \in \mathcal{M}\} \\ &= \widetilde{P}(y \in C) . \quad \underline{\text{(Consistency)}} \end{aligned}$$

Random sets: a is modeled by a random set with focal sets A^i and weights $m(A^i)$. Plausibility measure:

$$\widetilde{P}(y \in C) = \sum_{G(A^i) \cap C \neq \varnothing} m(A^i) .$$

Alternatively, a is the set of probablilities of the form $p = \sum m(A^i) p_i$ where each p_i belongs to the set $\mathcal{M}^i$ of probability measures on A^i. Upper probability:

$$\begin{aligned} \overline{P}(y \in C) &= \sup\{p(G^{-1}(C)) : p = \textstyle\sum m(A^i) p_i,\ p_i \in \mathcal{M}^i\} \\ &= \widetilde{P}(y \in C) . \quad \underline{\text{(Consistency)}} \end{aligned}$$

Sets of probability measures: a is modeled by a family of probability measures $\{p_\lambda : \lambda \in \Lambda\}$. If Λ is just a set, we put

$$\widetilde{P}(y \in C) = \sup\{p_\lambda\big(G^{-1}(C)\big) : \lambda \in \Lambda\} .$$

If Λ is itself a probability space with a (single) measure π, we define

$$P_\pi(y \in C) = \int_\Lambda p_\lambda\big(G^{-1}(C)\big)\, \mathrm{d}\pi(\lambda) .$$

If Λ is equipped with a family $\mathcal{M}$ of probability measures, the corresponding upper probability is defined by

$$\overline{P}(y \in C) = \sup\{P_\pi(y \in C) : \pi \in \mathcal{M}\} .$$

Then always

$$\overline{P}(y \in C) \le \widetilde{P}(y \in C) .$$

Equality holds if $\mathcal{M}$ contains all Dirac measures situated at the points of Λ. <u>(Consistency)</u>

Fuzzy sets: a is a fuzzy set with membership function $\mu_a : \mathbb{R} \to [0, 1]$. Possibility distribution on $\mathbb{R}$:

$$\mu_a(B) = \sup\{\mu_a(x) : x \in B\}$$

where $B \subset \mathbb{R}$. Wellknown:

$$\mu_a(x) = \sup\{p(\{x\}) : p \prec \mu_a\} .$$

Possibility degree of C under the function G:

$$\mu_{G(a)}(C) = \sup\{\mu_a(x) : x \in G^{-1}(y), y \in C\} .$$

Upper probability:

$$\begin{aligned} \overline{P}(y \in C) &= \sup\{p\big(G^{-1}(C)\big) : p \prec \mu_a\} \\ &= \mu_{G(a)}(C) . \quad \underline{\text{(Consistency)}} \end{aligned}$$

The multivariate case

The joint probability measures are of the form

$$P = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} m(A_1^i \times A_2^j) P^{ij}$$

and have marginals in $\mathcal{M}_1$, respectively $\mathcal{M}_2$; m is the joint weight function and P^{ij} are measures on $A_1^i \times A_2^j$. The results crucially depend on the notion of independence chosen – the question of interaction becomes decisive and requires a choice!

Notation: the marginals of P^{ij} in $\mathcal{M}_1^i$ and $\mathcal{M}_2^j$ will be denoted by $P_1^{i,ij}$ and $P_2^{j,ij}$, respectively.

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Strong independence:

- Joint weights $m(A_1^i \times A_2^j) = m_1(A_1^i) m_2(A_2^j)$.
- $P^{ij} = P_1^{i,ij} \otimes P_2^{j,ij}$.
- P^{ij} satisfies $P_1^{i,i1} = P_1^{i,i2} = \cdots = P_1^{i,in_2}$ and $P_2^{j,1j} = P_2^{j,2j} = \cdots = P_2^{j,n_1 j}$. ★

$\mathcal{M}_{\mathrm{S}} = \{P_1 \otimes P_2 : P_1 \in \mathcal{M}_1, P_2 \in \mathcal{M}_2\}$ is the set of joint probability measures.

Random set independence (Dempster-Shafer):

- Joint weights $m(A_1^i \times A_2^j) = m_1(A_1^i) m_2(A_2^j)$.
- P^{ij} are arbitrary with marginals in $\mathcal{M}_1^i$, $\mathcal{M}_2^j$.
- Condition ★ is dropped.

Unknown interaction (correlation):

- Weights $m(A_1^i \times A_2^j)$ are arbitrary subject to

$$\sum_{j=1}^{n_2} m(A_1^i \times A_2^j) = m_1(A_1^i), \ \sum_{i=1}^{n_1} m(A_1^i \times A_2^j) = m_2(A_2^j).$$

- Measures P^{ij} as for random set independence.

$\mathcal{M}_{\mathrm{U}} = \{P : P(\cdot \times \mathbb{R}) \in \mathcal{M}_1,\ P(\mathbb{R} \times \cdot) \in \mathcal{M}_2\}$.

Fuzzy set independence:

- In the case of consonant focal sets a random set a_k generates a fuzzy set $\tilde{a}_k$.
- The re-interpretation of the joint fuzzy set $\tilde{a} = \tilde{a}_1 \times \tilde{a}_2$ as a random set leads to the same joint focals as above, but in general with different weights.

Combination of probability measures parametrized by random sets: Here the set $\mathcal{M}_k^i$ is a set of probability measures $p_{a_k^i}$ parameterized by probability measures on the focal set A_k^i:

$$\mathcal{M}_k^i = \{P_\pi :\ P_\pi = \int_{A_k^i} p_{a_k^i}(\cdot)\, \mathrm{d}\pi(a_k^i),\ \pi(A_k^i) = 1\}.$$

For challenge problem 1 the lower and upper complementary cumulative distribution functions $\underline{F}(y_f) = \underline{P}(y \geq y_f)$ and $\overline{F}(y_f) = \overline{P}(y \geq y_f)$ are plotted. We use the indices S, R, U and F to indicate the different types of independence.

Relations between the types of independence.

- $\mathcal{M}_{\mathrm{S}} \subseteq \mathcal{M}_{\mathrm{R}} \subseteq \mathcal{M}_{\mathrm{U}}$ and $\mathcal{M}_{\mathrm{F}} \subseteq \mathcal{M}_{\mathrm{U}}$,
- $\underline{F}_{\mathrm{U}} \leq \underline{F}_{\mathrm{R}} \leq \underline{F}_{\mathrm{S}} \leq \overline{F}_{\mathrm{S}} \leq \overline{F}_{\mathrm{R}} \leq \overline{F}_{\mathrm{U}}$ and $\underline{F}_{\mathrm{U}} \leq \underline{F}_{\mathrm{F}} \leq \overline{F}_{\mathrm{F}} \leq \overline{F}_{\mathrm{U}}$.

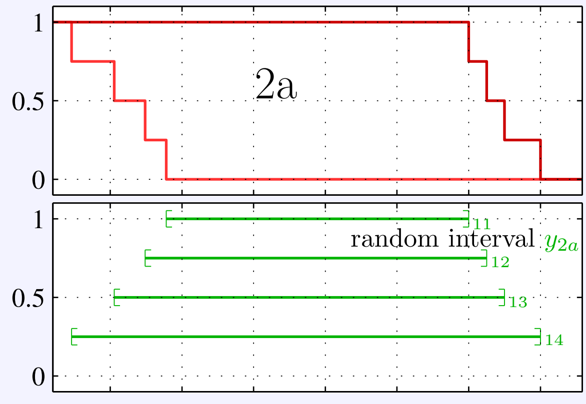
Problem 1: interval × interval.

$\underline{F}_{\mathrm{S}} = \underline{F}_{\mathrm{R}} = \underline{F}_{\mathrm{U}} = \underline{F}_{\mathrm{F}}$, $\overline{F}_{\mathrm{S}} = \overline{F}_{\mathrm{R}} = \overline{F}_{\mathrm{U}} = \overline{F}_{\mathrm{F}}$ and resulting interval $y_1 = [0.6922, 2]$, see problem 2.

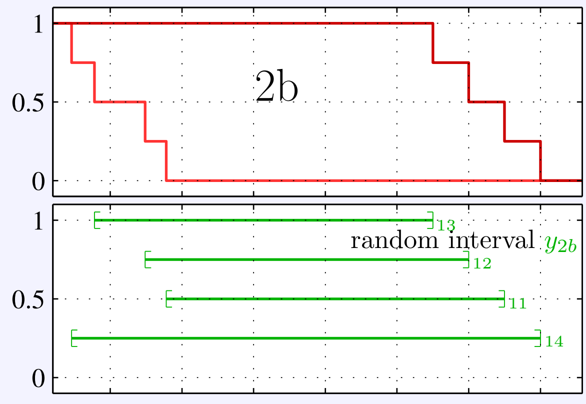
Problem 2: interval × random interval.

- The joint weights $m(A_1^i \times A_2^j)$ are unique.
- Due to the monotonicity of $(a_1 + a_2)^{a_1}$ in the a_2-direction, dropping the condition ★ does not change the result $\Longrightarrow$ $\underline{F}_{\mathrm{S}} = \underline{F}_{\mathrm{R}}$, $\overline{F}_{\mathrm{S}} = \overline{F}_{\mathrm{R}}$.

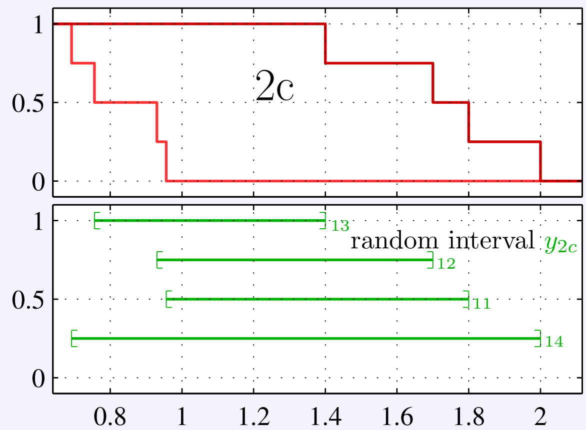
$\underline{F}_{\mathrm{S}} = \underline{F}_{\mathrm{R}} = \underline{F}_{\mathrm{U}} = \underline{F}_{\mathrm{F}}$, $\overline{F}_{\mathrm{S}} = \overline{F}_{\mathrm{R}} = \overline{F}_{\mathrm{U}} = \overline{F}_{\mathrm{F}}$



$\underline{F}_{\mathrm{S}} = \underline{F}_{\mathrm{R}} = \underline{F}_{\mathrm{U}}$, $\overline{F}_{\mathrm{S}} = \overline{F}_{\mathrm{R}} = \overline{F}_{\mathrm{U}}$



$\underline{F}_{\mathrm{S}} = \underline{F}_{\mathrm{R}} = \underline{F}_{\mathrm{U}}$, $\overline{F}_{\mathrm{S}} = \overline{F}_{\mathrm{R}} = \overline{F}_{\mathrm{U}}$



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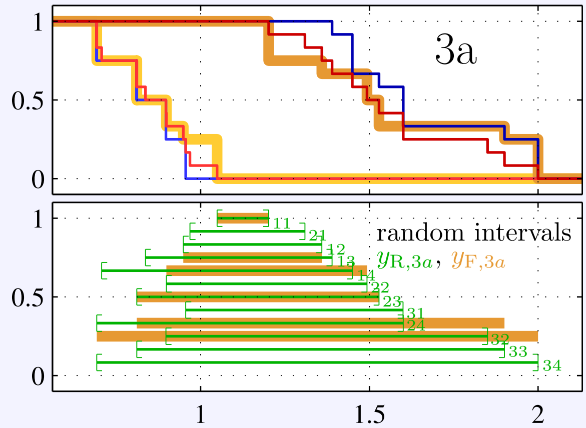
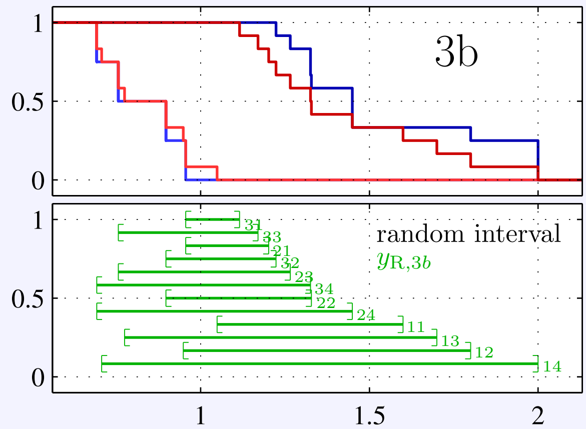
Problem 3:
random interval × random interval.

- By the same arguments as in problem 2: $\underline{F}_\mathrm{S} = \underline{F}_\mathrm{R}$ and $\overline{F}_\mathrm{S} = \overline{F}_\mathrm{R}$.
- The joint weights are not uniquely determined: Different results for random set independence, fuzzy set independence and unknown interaction.

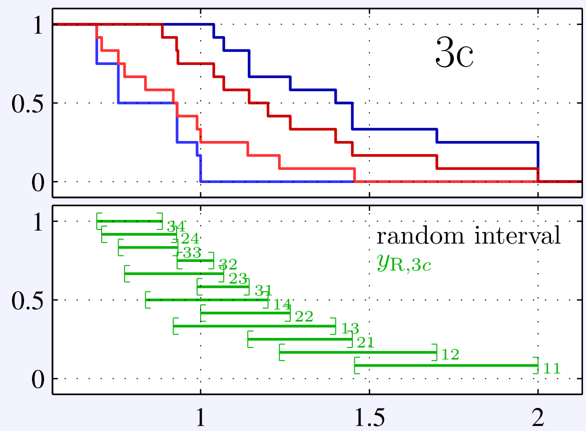
$\underline{F}_\mathrm{U} \le \underline{F}_\mathrm{S} = \underline{F}_\mathrm{R}$, $\overline{F}_\mathrm{R} = \overline{F}_\mathrm{S} \le \overline{F}_\mathrm{S}$, $\underline{F}_\mathrm{S}$, $\overline{F}_\mathrm{S}$



$\underline{F}_\mathrm{U} \le \underline{F}_\mathrm{S} = \underline{F}_\mathrm{R}$, $\overline{F}_\mathrm{R} = \overline{F}_\mathrm{S} \le \overline{F}_\mathrm{S}$



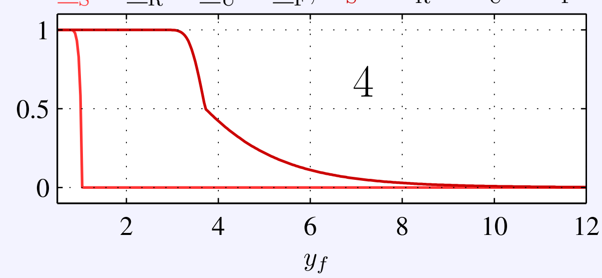
$\underline{F}_\mathrm{U} \le \underline{F}_\mathrm{S} = \underline{F}_\mathrm{R}$, $\overline{F}_\mathrm{R} = \overline{F}_\mathrm{S} \le \overline{F}_\mathrm{S}$



Problem 4:
interval × probability measure parametrized by an interval.

- By similar arguments as for problem 1 and 2: $\underline{F}_\mathrm{S} = \underline{F}_\mathrm{R} = \underline{F}_\mathrm{U} = \underline{F}_\mathrm{F}$, $\overline{F}_\mathrm{S} = \overline{F}_\mathrm{R} = \overline{F}_\mathrm{U} = \overline{F}_\mathrm{F}$.

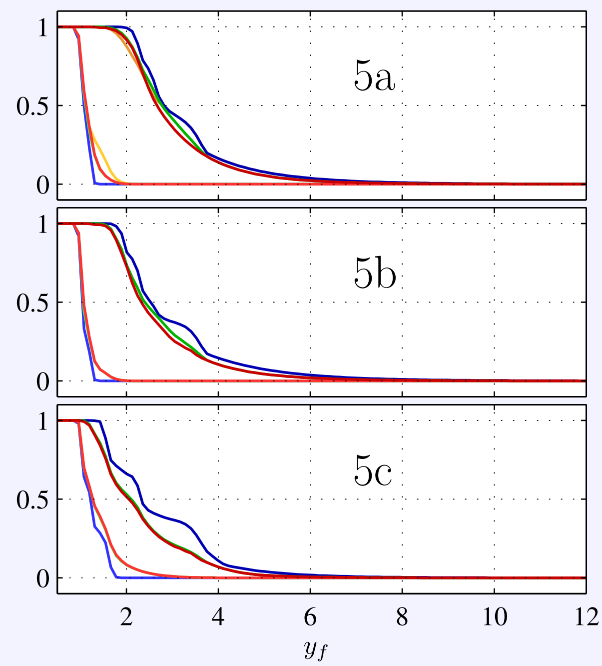
$\underline{F}_\mathrm{S} = \underline{F}_\mathrm{R} = \underline{F}_\mathrm{U} = \underline{F}_\mathrm{F}$, $\overline{F}_\mathrm{S} = \overline{F}_\mathrm{R} = \overline{F}_\mathrm{U} = \overline{F}_\mathrm{F}$



Problem 5:
random interval × probability measure parametrized by a random interval.

- $\overline{F}_\mathrm{S}(y_f) < \overline{F}_\mathrm{R}(y_f)$ holds for some y_f.

$\underline{F}_\mathrm{U} \le \underline{F}_\mathrm{R} \le \underline{F}_\mathrm{S}$, $\overline{F}_\mathrm{S} \le \overline{F}_\mathrm{R} \le \overline{F}_\mathrm{U}$, $\underline{F}_\mathrm{F}$, $\overline{F}_\mathrm{F}$



Problem 6: interval × probability measure.

$\underline{F}_\mathrm{S} = \underline{F}_\mathrm{R} = \underline{F}_\mathrm{U} = \underline{F}_\mathrm{F}$, $\overline{F}_\mathrm{S} = \overline{F}_\mathrm{R} = \overline{F}_\mathrm{U} = \overline{F}_\mathrm{F}$